

Graphs with degree constraints

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Abstract

Given a set \mathcal{D} of nonnegative integers, we derive the asymptotic number of graphs with a given number of vertices, edges, and such that the degree of every vertex is in \mathcal{D} . This generalizes existing results, such as the enumeration of graphs with a given minimum degree, and establishes new ones, such as the enumeration of Euler graphs, *i.e.* where all vertices have an even degree. Those results are derived using analytic combinatorics.

1 Introduction

1.1 Related works

The asymptotics of several families of simple graphs with degree constraints have been derived. Regular graphs, where all vertices have the same degree, have been enumerated by [Bender and Canfield \(1978\)](#), graphs with minimum degree at least δ by [Pittel and Wormald \(2003\)](#). An *Euler graph*, or *even graph*, is a graph where all vertices have an even degree. An exact formula for the number of such graphs, for a given number of vertices and without consideration of the number of edges, has been derived by [Robinson \(1969\)](#) and [Mallows and Sloane \(1975\)](#). In the present work, we generalize those results and derive the asymptotic number of graphs with degrees in any given set.

A similar problem has been addressed with probabilistic tools by the *configuration model*, introduced independently by [Bollobás \(1980\)](#) and [Wormald \(1978\)](#). This model inputs a distribution F on the degrees, and outputs a random multigraph where the degree of each vertex follows F . The main difference with the model analyzed in this article is that the number of edges in the configuration model is a random variable. The link between both models is discussed in [Section 4.1](#). For more information on the configuration model, we recommend the book of [van der Hofstad \(2014\)](#).

Other related problems include the enumeration of graphs with a given degree sequence ([Bender and Canfield \(1978\)](#)), the enumeration of symmetric matrices with nonnegative coefficients and constant row sum ([Chyzak et al. \(2005\)](#)), and the enumeration of graphs with degree parities, investigated by [Read and Robinson \(1982\)](#).

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1.2 Model and notations

A *multiset* is an unordered collection of objects, where repetitions are allowed. Sets are then multisets without repetitions. A *sequence* is an ordered multiset. We use the parenthesis notation (u_1, \dots, u_n) for sequences, and the brace notation $\{u_1, \dots, u_n\}$ for sets and multisets. Open real intervals are denoted by open square brackets $]a, b[$.

A *simple graph* G is a set $V(G)$ of labelled vertices and a set $E(G)$ of edges, where each edge is an unordered pair of distinct vertices. In a *multigraph*, the edges form a multiset and the vertices in an edge need not be distinct. An edge $\{v, w\}$ is a *loop* if $v = w$, a *multiple edge* if it has at least two occurrences in the multiset of edges, and a *simple edge* otherwise. Thus, the simple graphs are the multigraphs that contain neither loops nor multiple edges, i.e. that contain only simple edges. The set of multigraphs with n vertices and m edges is denoted by $\text{MG}_{n,m}$, and the subset of simple graphs by $\text{SG}_{n,m}$.

The *degree* of a vertex is defined as its number of occurrences in $E(G)$. In particular, a loop increases its degree by 2. The set of multigraphs from $\text{MG}_{n,m}$ where each vertex has its degree in a set \mathcal{D} is denoted by $\text{MG}_{n,m}^{(\mathcal{D})}$. The subset of simple graphs is $\text{SG}_{n,m}^{(\mathcal{D})}$. The set \mathcal{D} may be finite or infinite. We denote its generating function by

$$\text{Set}_{\mathcal{D}}(x) = \sum_{d \in \mathcal{D}} \frac{x^d}{d!}.$$

For any natural number i , $\mathcal{D} - i$ denotes the set $\{d - i \in \mathbb{Z}_{\geq 0} \mid d \in \mathcal{D}\}$. In particular, observe that $\text{Set}'_{\mathcal{D}}(x) = \text{Set}_{\mathcal{D}-1}(x)$. We also define the *valuation* $r = \min(\mathcal{D})$ and *periodicity* $p = \gcd\{d_1 - d_2 \mid d_1, d_2 \in \mathcal{D}\}$ of the set \mathcal{D} (by convention, the periodicity is infinite when $|\mathcal{D}| = 1$).

2 Main Theorem and applications

Our main result is an asymptotic expression for the number of graphs in $\text{SG}_{n,m}^{(\mathcal{D})}$, when the number m of edges grows linearly with the number n of vertices.

Theorem 1. *Assume \mathcal{D} contains at least two integers, has valuation $r = \min\{d \in \mathcal{D}\}$ and periodicity $p = \gcd\{d_1 - d_2 \mid d_1, d_2 \in \mathcal{D}\}$. Let m, n denote two integers tending to infinity, such that $2m/n$ stays in a fixed compact interval of $]r, \max(\mathcal{D})[$ and p divides $2m - rn$, then the number of simple graphs in $\text{SG}_{n,m}^{(\mathcal{D})}$ is*

$$\frac{(2m)!}{2^m m!} \frac{\text{Set}_{\mathcal{D}}(\zeta)^n}{\zeta^{2m}} \frac{p}{\sqrt{2\pi n \zeta \phi'(\zeta)}} e^{-W_{\frac{n}{m}}(\zeta)^2 - W_{\frac{n}{m}}(\zeta)} (1 + O(n^{-1})),$$

where $\phi(x) = \frac{x \text{Set}_{\mathcal{D}-1}(x)}{\text{Set}_{\mathcal{D}}(x)}$, ζ is the unique positive solution of $\phi(\zeta) = \frac{2m}{n}$, and $W_{\frac{n}{m}}(x) = \frac{n}{4m} \frac{x^2 \text{Set}_{\mathcal{D}-2}(x)}{\text{Set}_{\mathcal{D}}(x)}$. If p does not divide $2m - rn$, if $2m/n < r$ or if $2m/n > \max(\mathcal{D})$, then $\text{SG}_{n,m}^{(\mathcal{D})}$ is empty.

When $\mathcal{D} = \mathbb{Z}_{\geq 0}$, the degrees are not constrained, so $\text{SG}_{n,m}^{(\mathcal{D})} = \text{SG}_{n,m}$. Using Stirling formula, it can indeed be checked that $\binom{n}{m}$, the total number of simple graphs with n vertices and m edges, is asymptotically equal to the result of Theorem 1

$$\frac{n^{2m}}{2^m m!} \frac{(2m)!}{(2m)^{2m} e^{-2m} \sqrt{2\pi 2m}} e^{-\left(\frac{m}{n}\right)^2 - \frac{m}{n}} (1 + O(n^{-1})).$$

Pittet and Wormald (2003) have derived the asymptotics of simple graphs with minimum degree at least δ . They used probabilistic and analytic elementary tools, in a sophisticated way. In the present paper, we have addressed the enumeration of a broader family of graphs with degree constraints, using more powerful tools (analytic combinatorics). For graphs with minimum degree at least δ , the asymptotics derived in Theorem 1, for $\mathcal{D} = \mathbb{Z}_{\geq \delta}$, matches their result.

Euler graphs are simple graphs where each vertex has an even degree. An exact, but complicated, formula for the number of such graphs, for given number of vertices and without consideration of the number of edges,

has been derived by [Robinson \(1969\)](#) and [Mallows and Sloane \(1975\)](#). Applying Theorem 1, we are now able to derive the asymptotic number of Euler graphs with n vertices and m edges, when $2m/n$ stays in a fixed compact interval of $\mathbb{R}_{>0}$

$$|\text{SG}_{n,m}^{(\text{even})}| = \frac{(2m)!}{2^m m!} \frac{\cosh(\zeta)^n}{\zeta^{2m}} \frac{2}{\sqrt{2\pi n \zeta \phi'(\zeta)}} e^{-\left(\frac{n}{4m}\zeta^2\right)^2 - \frac{n}{4m}\zeta^2} (1 + O(n^{-1})),$$

where $\phi(x) = x \tanh(x)$ and $\tanh(\zeta) = 2m/n$.

3 Proof of the result

In this section we provide a proof for Theorem 1. The proof of all lemmas and theorems are moved to the appendix.

3.1 Preliminaries

3.1.1 Multigraph model

The main model of random multigraphs with n vertices and m edges is the *multigraph process*, analyzed by [Flajolet et al. \(1989\)](#) and [Janson et al. \(1993\)](#). It samples uniformly and independently $2m$ vertices $(v_1, v_2, \dots, v_{2m})$ in $\{1, \dots, n\}$, and outputs a multigraph with set of vertices $\{1, \dots, n\}$ and set of edges

$$\{\{v_{2i-1}, v_{2i}\} \mid 1 \leq i \leq m\}.$$

Given a simple or multi graph, one can order the set of edges and the vertices in each edge. The result is a sequence of ordered pairs of vertices, that we call an *ordering* of G . Let $\text{orderings}(G)$ denote the number of such orderings. For example, the multigraph on 2 vertices with edges $\{\{1, 1\}, \{1, 2\}, \{1, 2\}\}$ has 12 orderings, amongst them $((1, 2), (1, 1), (2, 1))$. For simple graphs, the number of orderings is equal to $2^m m!$, because each edge has two possible orientations and all edges can be permuted. For non-simple multigraphs, orderings is smaller. [Flajolet et al. \(1989\)](#) and [Janson et al. \(1993\)](#) introduced the *compensation factor* $\kappa(G)$ of a multigraph G with m edges, defined as

$$\kappa(G) = \frac{\text{orderings}(G)}{2^m m!}.$$

The compensation factor of a multigraph is 1 if and only if it is simple.

Observe that in the random distribution induced by the multigraph process, each multigraph receives a probability proportional to its compensation factor. Therefore, when the output of the multigraph process is constrained to be a simple graph, the sampling becomes uniform on $\text{SG}_{n,m}$. The *total weight* of a family \mathcal{F} of multigraphs is the sum of their compensation factors. For example, the total weight of $\text{MG}_{n,m}$ is equal to $\frac{n^{2m}}{2^m m!}$. When \mathcal{F} contains only simple graphs, its total weight is equal to its cardinality.

3.1.2 Analytic tools

Our tool for the analysis of graphs with degree constraints is *analytic combinatorics*, as presented by [Flajolet and Sedgewick \(2009\)](#). Its principle is to associate to the combinatorial family studied its *generating function*. The asymptotics of the family is then linked to the analytic behavior of this function.

In the analysis of a graphs family \mathcal{F} with analytic combinatorics, the main difficulty is the fast growth of its cardinality, which often implies a zero radius of convergence for the corresponding generating function

$$\sum_{G \in \mathcal{F}} w^{|E(G)|} \frac{z^{|V(G)|}}{|V(G)|!}.$$

This feature drastically reduces the number of tools from complex analysis that can be applied. Graphs with degree constraints are no exception, but our approach completely avoid this classic issue. In fact, the only analytic tool we use is the following lemma, a variant of ([Flajolet and Sedgewick, 2009](#), Theorem VIII.8).

Lemma 2. Consider a non-monomial series $B(z)$ with nonnegative coefficients, analytic on \mathbb{C} , with valuation $r = \min\{n \mid [z^n]B(z) \neq 0\}$ and periodicity $p = \gcd\{n \mid [z^{n-r}]B(z) \neq 0\}$. Let $\phi(z)$ denote the function $\frac{zB'(z)}{B(z)}$, and K a compact interval of the open interval $]r, \lim_{x \rightarrow \infty} \phi(x)[$. Let N, n denote two integers tending to infinity while N/n stays in K , and let ζ denote the unique positive solution of $\phi(\zeta) = N/n$. Finally, consider a compact Y and a function $A(y, z)$, \mathcal{C}^2 on $Y \times \mathbb{C}$, such that for all y in Y , the function $z \mapsto A(y, z)$ is analytic on \mathbb{C} and $A(y, \zeta^p)$ is nonzero. Then we have, uniformly for N/n in K and y in Y ,

$$[z^N]A(y, z^p)B(z)^n = \begin{cases} \frac{pA(y, \zeta^p)}{\sqrt{2\pi n\zeta\phi'(\zeta)}} \frac{B(\zeta)^n}{\zeta^N} (1 + O(n^{-1})) & \text{if } p \text{ divides } N - nr, \\ 0 & \text{otherwise.} \end{cases}$$

3.2 Multigraphs with degree constraints

The work of [Flajolet et al. \(1989\)](#) and [Janson et al. \(1993\)](#) demonstrates that multigraphs are more suitable to the analytic combinatorics approach than simple graphs. Moreover, the results on multigraphs can usually be extended to simple graphs. Following this observation, multigraphs are analyzed in this section, before turning so simple graphs in Section [3.3](#).

3.2.1 Exact and asymptotic enumeration

We derive an exact expression for the number of multigraphs with degree constraints in Theorem [3](#), then translates it into an asymptotics in Theorem [4](#).

Theorem 3. The total weight of all multigraphs in $\text{MG}_{n,m}^{(\mathcal{D})}$ is

$$\sum_{G \in \text{MG}_{n,m}^{(\mathcal{D})}} \kappa(G) = \frac{(2m)!}{2^m m!} [x^{2m}] \text{Set}_{\mathcal{D}}(x)^n.$$

The proof of this theorem is elementary by the definition of the compensation factor. Now applying Lemma [2](#) to the exact expression, we derive the asymptotics of multigraphs with degree constraints. Let us first eliminate three simple cases.

- When \mathcal{D} contains only one integer $\mathcal{D} = \{d\}$, $\text{MG}_{n,m}^{(\mathcal{D})}$ is the set of d -regular multigraphs. The total weight of $\text{MG}_{n,m}^{(\mathcal{D})}$ is then 0 if $2m \neq nd$, and $\frac{(2m)!}{2^m m! d^n}$ otherwise.
- The sum of the degrees of the vertices is equal to $2m$, so $\text{MG}_{n,m}^{(\mathcal{D})}$ is empty when $2m/n < \min(\mathcal{D})$ or $2m/n > \max(\mathcal{D})$.
- The periodicity p of \mathcal{D} is equal to $\gcd\{d - r \mid d \in \mathcal{D}\}$. For each vertex v of a multigraph from $\text{MG}_{n,m}^{(\mathcal{D})}$, it follows that p divides $\deg(v) - r$. By summation over all vertices, we conclude that if p does not divide $2m - nr$, then the set $\text{MG}_{n,m}^{(\mathcal{D})}$ is empty.

The two last points obviously hold for $\text{SG}_{n,m}^{(\mathcal{D})}$.

Theorem 4. Consider a set $\mathcal{D} \subset \mathbb{Z}_{\geq 0}$ of size at least 2. Let $r = \min(\mathcal{D})$ denote its valuation and $p = \gcd\{d_1 - d_2 \mid d_1, d_2 \in \mathcal{D}\}$ its periodicity. Let m, n denote two integers tending to infinity, such that $2m/n$ stays in a fixed compact interval of the open interval $]r, \max(\mathcal{D})[$, and p divides $2m - rn$, then the total weight of $\text{MG}_{n,m}^{(\mathcal{D})}$ is equal to

$$\sum_{G \in \text{MG}_{n,m}^{(\mathcal{D})}} \kappa(G) = \frac{(2m)!}{2^m m!} \frac{p}{\sqrt{2\pi n\zeta\phi'(\zeta)}} \frac{\text{Set}_{\mathcal{D}}(\zeta)^n}{\zeta^{2m}} (1 + O(n^{-1}))$$

where $\phi(x) = \frac{x \text{Set}_{\mathcal{D}-1}(x)}{\text{Set}_{\mathcal{D}}(x)}$ and ζ is the unique positive solution of $\phi(\zeta) = \frac{2m}{n}$.

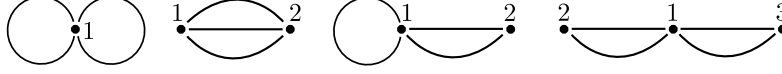


Figure 1: Four examples of multigraphs from $\text{MG}_{n,m}^{(\mathcal{D},0)}$.

3.2.2 Typical multigraphs with degree constraints

Let us recall that an edge is *simple* if it is neither a loop nor a multiple edge. Before turning to the enumeration of simple graphs with degree constraints, we first describe the behavior of non-simple edges in a typical multigraph from $\text{MG}_{n,m}^{(\mathcal{D})}$. No proofs are given here, as stronger results will be derived later.

Using random sampling, we observe that in most of the multigraphs from $\text{MG}_{n,m}^{(\mathcal{D})}$, all non-simple edges have low multiplicity and are well separated. This motivates the following definition. A multigraph from $\text{MG}_{n,m}^{(\mathcal{D})}$ is in $\text{MG}_{n,m}^{(\mathcal{D},*)}$ if all its non-simple edges are loops or double edges, and each vertex belongs to at most one loop or (exclusive) one double edge. Let $|E|_e$ denote the number of occurrences of the element e in the multiset E . Formally, $\text{MG}_{n,m}^{(\mathcal{D},*)}$ is characterized as the set of multigraphs G from $\text{MG}_{n,m}^{(\mathcal{D})}$ such that for all vertices u, v, w , we have

$$\begin{aligned} |E(G)|_{\{v,v\}} &\leq 1, & |E(G)|_{\{u,v\}} = |E(G)|_{\{v,w\}} = 2 &\implies u = w, \\ |E(G)|_{\{v,w\}} &\leq 2, & \{v, v\} \in E(G) &\implies \forall w, |E(G)|_{\{v,w\}} \leq 1. \end{aligned}$$

The complementary set, $\text{MG}_{n,m}^{(\mathcal{D})} \setminus \text{MG}_{n,m}^{(\mathcal{D},*)}$, is denoted by $\text{MG}_{n,m}^{(\mathcal{D},0)}$, and illustrated in Figure 1.

3.3 Simple graphs with degree constraints

We introduce the notation $\text{SG}_{n,m}^{(\mathcal{D})}$ for the set of simple graphs with n vertices, m edges and all degrees in \mathcal{D} , *i.e.* multigraphs from $\text{MG}_{n,m}^{(\mathcal{D})}$ that contain neither loops nor multiple edges. The enumeration of simple graphs with degree constraints is derived in Theorem 1. First, in Section 3.3.2, we describe an inclusion-exclusion process that outputs $|\text{SG}_{n,m}^{(\mathcal{D})}|$ when applied to $\text{MG}_{n,m}^{(\mathcal{D},*)}$. In Section 3.3.3, this process is then applied to $\text{MG}_{n,m}^{(\mathcal{D})}$, and the error introduced is proven to be negligible in Section 3.3.4.

In order to forbid loops and multiple edges in multigraphs from $\text{MG}_{n,m}^{(\mathcal{D})}$, we introduce the notion of *marked multigraphs*.

3.3.1 Marked multigraphs

A *marked multigraph* G is a triplet $(V(G), E(G), \bar{E}(G))$, where $V(G)$ denotes the set of vertices, $E(G)$ the multiset of *normal edges*, and $\bar{E}(G)$ the multiset of *marked edges*, where both normal and marked edges are unordered pairs of vertices. We say that a marked multigraph G belongs to a family \mathcal{F} of (unmarked) multigraphs if the unmarked multigraph $(V(G), E(G) \cup \bar{E}(G))$ is in \mathcal{F} .

We now extend to marked multigraphs the definitions of degree, orderings and compensation factors, introduced for multigraphs in Section 3.1. The *degree* of a vertex from a marked multigraph G is equal to its number of occurrences in the multiset $E(G) \cup \bar{E}(G)$. An *ordering* of a marked multigraph G with $m = |E(G)| + |\bar{E}(G)|$ edges is a sequence

$$S = ((v_1, w_1, t_1), \dots, (v_m, w_m, t_m))$$

from $(V(G) \times V(G) \times \{0, 1\})^m$ such that the multiset $\{\{v_i, w_i\} \mid (v_i, w_i, 0) \in S\}$ is equal to $E(G)$, and the multiset $\{\{v_i, w_i\} \mid (v_i, w_i, 1) \in S\}$ is equal to $\bar{E}(G)$. The number of orderings of a given marked multigraph G is denoted by $\text{orderings}(G)$, and its *compensation factor* is

$$\kappa(G) = \frac{\text{orderings}(G)}{2^m m!}.$$

For example, consider the marked multigraph G with

$$V(G) = \{1, 2\}, \quad E(G) = \{\{1, 2\}\}, \quad \bar{E}(G) = \{\{1, 2\}, \{1, 2\}\}.$$

Its number of orderings is 24, and therefore its compensation factor is $\kappa(G) = 1/2$, whereas it is $1/6$ for G without the marks,

$$V(G) = \{1, 2\}, \quad E(G) = \{\{1, 2\}, \{1, 2\}, \{1, 2\}\}.$$

In the following, we will consider families of marked multigraphs where the marked edges are loops or multiple edges. Given a marked multigraph G , then $\ell(G)$ denotes the number of loops in $\bar{E}(G)$, and $k(G)$ the number of distinct edges from $\bar{E}(G)$ that are not loops. The generating function of a family \mathcal{F} of marked multigraphs is

$$F(u, v) = \sum_{G \in \mathcal{F}} \kappa(G) u^{k(G)} v^{\ell(G)}.$$

3.3.2 Inclusion-exclusion process

In this section, we build an operator `Marked` that inputs a family of multigraphs and outputs a family of marked multigraphs. It is designed so that the asymptotics of its generating function $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(u, v)$ is linked to the asymptotics of $|\text{SG}_{n,m}^{(\mathcal{D})}|$. In order to justify the construction, we first introduce the operators $\text{Marked}^{(1)}$ and $\text{Marked}^{(2)}$.

First marking. If we could mark all loops and multiple edges from $\text{MG}_{n,m}^{(\mathcal{D})}$, the enumeration of simple graphs with degree constraints would be easy. Indeed, given a family \mathcal{F} of multigraphs, let $\text{Marked}_{\mathcal{F}}^{(1)}$ denote the marked multigraphs from \mathcal{F} with all loops and multiple edges marked. Since the simple graphs are the multigraphs that have neither loops nor multiple edges, we have

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}^{(1)}(0, 0) = \sum_{G \in \text{MG}_{n,m}^{(\mathcal{D})}} \kappa(G) 0^{k(G)} 0^{\ell(G)} = \sum_{G \in \text{SG}_{n,m}^{(\mathcal{D})}} \kappa(G),$$

which is equal to $|\text{SG}_{n,m}^{(\mathcal{D})}|$, because simple graphs have a compensation factor equal to 1. Unfortunately, we do not have a description of this family in the symbolic method formalism.

Second marking. The inclusion-exclusion principle advises us to mark *some* of the non-simple edges. Let $\text{Marked}_{\mathcal{F}}^{(2)}$ denote the set of marked multigraphs G from \mathcal{F} such that each edge from $\bar{E}(G)$ is either a loop, or has multiplicity at least 2 in $\bar{E}(G)$ and does not belong to $E(G)$. This construction implies the relation

$$\text{Marked}_{\mathcal{F}}^{(2)}(u, v) = \text{Marked}_{\mathcal{F}}^{(1)}(u + 1, v + 1),$$

and therefore

$$|\text{SG}_{n,m}^{(\mathcal{D})}| = \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}^{(2)}(-1, -1).$$

The natural idea to build a marked multigraph G from $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}^{(2)}$ is to first choose some loops and multiple edges to put in $\bar{E}(G)$, then complete $E(G)$ with unmarked edges, which may well form other loops and multiple edges, in a way that ensures $G \in \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}^{(2)}$. However, the description of the set of marked edges is complicated, because of the numerous possible intersection patterns.

Third marking. We have seen in Section 3.2.2 that in most of the multigraphs from $\text{MG}_{n,m}^{(\mathcal{D})}$, non-simple edges do not intersect. This motivates the following definition. Given a set \mathcal{F} of multigraphs, let $\text{Marked}(\mathcal{F})$ denote the set of marked multigraphs from \mathcal{F} such that each vertex is in exactly one of the following cases:

- the vertex belongs to no marked edge,
- the vertex belongs to one marked loop and no other marked edge,
- the vertex belongs to two identical marked edges and no other marked edge.

Therefore, each marked edge is a loop of multiplicity 1 or a double edge. This marking process links the multigraphs from $\text{MG}_{n,m}^{(\mathcal{D},*)}$, defined in Section 3.2.2, to the simple graphs with degree constraints.

Lemma 5. *The value $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},*)}}(-1, -1)$ is equal to the number of simple graphs in $\text{SG}_{n,m}^{(\mathcal{D})}$.*

Applying the operator Marked to the decomposition

$$\text{MG}_{n,m}^{(\mathcal{D})} = \text{MG}_{n,m}^{(\mathcal{D},*)} \uplus \text{MG}_{n,m}^{(\mathcal{D},0)},$$

we find

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(u, v) = \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},*)}}(u, v) + \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},0)}}(u, v)$$

which implies, after evaluation at $(u, v) = (-1, -1)$ and reordering of the terms,

$$|\text{SG}_{n,m}^{(\mathcal{D})}| = \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(-1, -1) - \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},0)}}(-1, -1).$$

We compute the asymptotics of $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(-1, -1)$ in Section 3.3.3, and prove that $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},0)}}(-1, -1)$ is negligible in Section 3.3.4.

3.3.3 Application of the inclusion-exclusion process to all multigraphs with degree constraints

We start with an exact expression of $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(u, v)$ in Lemma 6, then derive its asymptotics in Lemma 8.

Lemma 6. *We have the formal equality*

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(u, v) = \frac{(2m)!}{2^m m!} [x^{2m}] \left(\sum_{k, \ell \geq 0} a_{n,m,2k+\ell} \frac{(u W_{\frac{n}{m}}(x)^2)^k}{k!} \frac{(v W_{\frac{n}{m}}(x))^\ell}{\ell!} \right) \text{Set}_{\mathcal{D}}(x)^n,$$

where $a_{n,m,j} = 0$ when j is greater than $\min(n, m)$, otherwise

$$a_{n,m,j} = \frac{n!}{(n-j)! n^j} \frac{m!}{(m-j)! m^j} \frac{(2m-2j)!(2m)^{2j}}{(2m)!},$$

$$W_{\frac{n}{m}}(x) = \frac{n}{4m} \frac{x^2 \text{Set}_{\mathcal{D}-2}(x)}{\text{Set}_{\mathcal{D}}(x)}.$$

The proof is constructive by considering all the disjoint sets of vertices where we can put a loop or a double edges. We observe that when $2k + \ell$ is fixed while n, m tends to infinity, then $a_{n,m,2k+\ell}$ tends to 1. The double sum can then be approximated by an exponential, and it is tempting to conclude

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(u, v) \sim [x^{2m}] e^{u W_{\frac{n}{m}}(x)^2 + v W_{\frac{n}{m}}(x)} \text{Set}_{\mathcal{D}}(x)^n.$$

The next lemma formalize this intuition. A multivariate generating function $f(x_1, \dots, x_n)$ is said to *dominate coefficient-wise* another series $g(x_1, \dots, x_n)$ if for all $k_1, \dots, k_n \geq 0$,

$$\left| [x_1^{k_1} \cdots x_n^{k_n}] g(x_1, \dots, x_n) \right| \leq [x_1^{k_1} \cdots x_n^{k_n}] f(x_1, \dots, x_n).$$

Lemma 7. When m/n stays in a fixed compact interval of $\mathbb{R}_{>0}$, there is an entire bivariate analytic function $C(u, v)$ such that, for n large enough, $\frac{1}{n}C(u, v)$ dominates coefficient-wise

$$e^{u+v} - \sum_{k, \ell \geq 0} a_{n, m, 2k+\ell} \frac{u^k}{k!} \frac{v^\ell}{\ell!}. \quad (1)$$

We can now derive the asymptotics of $\text{Marked}_{\text{MG}_{n, m}^{(\mathcal{D})}}(u, v)$. As observed in the discussion preceding Theorem 4, the result is trivial when \mathcal{D} contains only one integer, when $2m/n$ is outside $[\min(\mathcal{D}), \max(\mathcal{D})]$ and when p does not divide $2m - \min(\mathcal{D})n$.

Lemma 8. Assume \mathcal{D} has size at least 2, valuation r and periodicity p . Let m, n denote two integers tending to infinity, such that $2m/n$ stays in a fixed compact interval of $]r, \max(\mathcal{D})[$ and p divides $2m - rn$. When u, v stay in a fixed compact, then

$$\text{Marked}_{\text{MG}_{n, m}^{(\mathcal{D})}}(u, v) = \frac{(2m)!}{2^m m!} \frac{\text{Set}_{\mathcal{D}}(\zeta)^n}{\zeta^{2m}} \frac{p}{\sqrt{2\pi n \zeta \phi'(\zeta)}} e^{u W_{\frac{n}{m}}(\zeta)^2 + v W_{\frac{n}{m}}(\zeta)} (1 + O(n^{-1})), \quad (2)$$

where $W_{\frac{n}{m}}(x) = \frac{n}{4m} \frac{x^2 \text{Set}_{\mathcal{D}-2}(x)}{\text{Set}_{\mathcal{D}}(x)}$, $\phi(x) = \frac{x \text{Set}_{\mathcal{D}-1}(x)}{\text{Set}_{\mathcal{D}}(x)}$ and $\phi(\zeta) = \frac{2m}{n}$.

The proof is a consequence of Lemma 2, Lemma 6 and Lemma 7.

3.3.4 Negligible marked multigraphs

Recall that $\text{MG}_{n, m}^{(\mathcal{D}, 0)}$ denotes the set $\text{MG}_{n, m}^{(\mathcal{D})} \setminus \text{MG}_{n, m}^{(\mathcal{D}, *)}$. In Lemma 10, we prove that $\text{Marked}_{\text{MG}_{n, m}^{(\mathcal{D}, 0)}}(-1, -1)$ is negligible. To do so, we first bound $\text{Marked}_R(1, 1)$ for a family R of marked multigraphs from $\text{MG}_{n, m}^{(\mathcal{D})}$ with mandatory edges.

Lemma 9. Let e_1, \dots, e_j denote j edges on the set of vertices $\{1, \dots, n\}$, and R the set of multigraphs from $\text{MG}_{n, m}^{(\mathcal{D})}$ that contain those edges, with multiplicities (i.e. an edge with k occurrences in the list has at least k occurrences in the multiset of edges of the multigraph)

$$R = \left\{ G \in \text{MG}_{n, m}^{(\mathcal{D})} \mid \forall 1 \leq i \leq j, e_i \in E(G) \text{ with multiplicities} \right\}.$$

Assume \mathcal{D} contains at least two integers and has valuation r . Let m, n denote two integers tending to infinity, such that $2m/n$ stays in a fixed compact interval of $]r, \max(\mathcal{D})[$, then

$$\text{Marked}_R(1, 1) = O(n^{-j} \text{Marked}_{\text{MG}_{n, m}^{(\mathcal{D})}}(1, 1)).$$

Figure 1 displays four multigraphs from $\text{MG}_{n, m}^{(\mathcal{D}, 0)}$. Actually, any multigraph from $\text{MG}_{n, m}^{(\mathcal{D}, 0)}$ contains one of those four graphs as a subgraph, and this property can be described in terms of mandatory edges. In the following lemma, we use this fact to bound $\text{Marked}_{\text{MG}_{n, m}^{(\mathcal{D}, 0)}}(-1, -1)$.

Lemma 10. Assume \mathcal{D} contains at least two integers, has valuation r and periodicity p . Let m, n denote two integers tending to infinity, such that $2m/n$ stays in a fixed compact interval of $]r, \max(\mathcal{D})[$, and p divides $2m - nr$, then

$$\text{Marked}_{\text{MG}_{n, m}^{(\mathcal{D}, 0)}}(-1, -1) = O\left(n^{-1} \text{Marked}_{\text{MG}_{n, m}^{(\mathcal{D})}}(-1, -1)\right).$$

The intuition supporting this proof is that a multigraph G belongs to $\text{MG}_{n, m}^{(\mathcal{D}, 0)}$ if and only if it contains a vertex v that is in one of the four configurations depicted in Figure 1. According to Lemma 9, multigraphs from $\text{MG}_{n, m}^{(\mathcal{D})}$ that contain those subgraphs have a negligible total weight. Now we have all the ingredients to prove Theorem 1.

Proof of Theorem 1. In Lemma 5, we have proven that the number of simple graphs in $\text{SG}_{n,m}^{(\mathcal{D})}$ is equal to $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},*)}}(-1, -1)$. By a set manipulation, this quantity can be rewritten

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(-1, -1) - \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},0)}}(-1, -1),$$

where $\text{MG}_{n,m}^{(\mathcal{D},0)} = \text{MG}_{n,m}^{(\mathcal{D})} \setminus \text{MG}_{n,m}^{(\mathcal{D},*)}$. Replacing the second term with the result of Lemma 10, we obtain

$$|\text{SG}_{n,m}^{(\mathcal{D})}| = \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(-1, -1)(1 + O(n^{-1})).$$

Finally, the asymptotics of $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(-1, -1)$ has been derived in Lemma 8. \square

4 Random generation

In order to keep a combinatorial interpretation, we focused on generating functions $\text{Set}_{\mathcal{D}}(x)$ with coefficients in $\{0, 1\}$. Our results hold more generally for any generating function $D(x)$ with nonnegative coefficients and large enough radius of convergence (so that the saddle-point from Lemma 2 is well defined). Multigraphs are then counted with a weight that depends of the degrees of their vertices

$$\text{weight}(G) = \kappa(G) \prod_{v \in V(G)} \deg(v)! [x^{\deg(v)}] D(x).$$

The present work has been guided by experiments on large random graphs with degree constraints. We used exact and Boltzmann sampling (Duchon et al. (2004)). Observe that to build a random simple graph from $\text{SG}_{n,m}^{(\mathcal{D})}$, one can sample multigraphs from $\text{MG}_{n,m}^{(\mathcal{D})}$ and reject until the multigraph is simple. As a consequence of Theorem 1, the expected number of rejections is $e^{-W_{\frac{n}{m}}(\zeta)^2 - W_{\frac{n}{m}}(\zeta)}$ (using the notations of the theorem).

4.1 Boltzmann sampling

The construction of the Boltzmann algorithm is straightforward from Theorem 3. To build a random multigraph with degrees in \mathcal{D} , n vertices and approximately m edges, the algorithm first computes a positive value x , according to the number of edges targeted. It then draws independently n integers (d_1, \dots, d_n) , following the law

$$\mathbb{P}(d) = \frac{([z^d] D(z)) x^d}{D(x)} \quad (3)$$

with $D(x) = \text{Set}_{\mathcal{D}}(x)$. If their sum is odd, a new sequence is drawn. Otherwise, the algorithm outputs a random multigraph with sequence of degrees (d_1, \dots, d_n) . To do so, as in the configuration model (Bollobás (1980), Wormald (1978)), each vertex v_i receives d_i half-edges, and a random pairing on the half-edges is drawn uniformly.

Therefore, the random distribution induced on multigraphs by the Boltzmann sampling algorithm is identical to the configuration model. Conversely, given a probability distribution on $\mathbb{Z}_{\geq 0}$, one can choose $D(x)$ so that the distribution is equal to the one described by Equation (3). Thus, we expect random multigraphs from the configuration model and multigraphs with degree constraints to share many statistical properties.

4.2 Recursive method

For the sampling of a multigraph in $\text{MG}_{n,m}^{(\mathcal{D})}$, the generator first draws a sequence of degrees, and then performs a random pairing of half-edges, as in configuration model and the Boltzmann sampler. Each

sequence (d_1, \dots, d_n) from \mathcal{D}^n is drawn with weight $\prod_{v=1}^n 1/(d_v)!$. In the first step, we use dynamic programming to precompute the values $(S_{i,j})_{0 \leq i \leq n, 0 \leq j \leq 2m}$, sums of the weights of all the sequences of i degrees that sum to j

$$S_{i,j} = \sum_{\substack{d_1, \dots, d_i \in \mathcal{D} \\ d_1 + \dots + d_i = j}} \prod_{v=1}^i \frac{1}{d_v!},$$

using the initial conditions and the recursive expression

$$S_{i,j} = \begin{cases} 1 & \text{if } (i,j) = (0,0), \\ 0 & \text{if } i = 0 \text{ and } j \neq 0, \text{ or if } j < 0, \\ \sum_{d \in \mathcal{D}} \frac{S_{i-1, j-d}}{d!} & \text{otherwise.} \end{cases}$$

After this precomputation, we generate the sequence of degrees as follows: first we sample the last degree d_n of the sequence according to the distribution

$$\mathbb{P}(d_n = d) = \frac{S_{n-1, 2m-d}}{d! S_{n, 2m}},$$

then we recursively generate the remaining sequence (d_1, \dots, d_{n-1}) , which must sum to $2m - d_n$. Once the sequence of degrees is computed, we generate a random pairing on the corresponding half-edges.

5 Forthcoming research

The results presented can be extended in several ways. The case where $2m/n$ tends to $\min(\mathcal{D})$ or $\max(\mathcal{D})$ could be considered. For example, [Pittel and Wormald \(2003\)](#) have derived, using elementary tools, the asymptotics of graphs with a lower bound on the minimum degree when $m = O(n \log(n))$. This extension would only require to adjust the saddle-point method from [Lemma 2](#).

We have also derived results on the enumeration of graphs where the degree sets vary with the vertices. The model inputs an infinite sequence of sets $(\mathcal{D}_1, \mathcal{D}_2, \dots)$ and output graphs where each vertex v has its degree in \mathcal{D}_v . The techniques presented in this paper can be extended to this case, if some technical conditions are satisfied, such as the convergence of the series $n^{-1} \sum_{v \geq 1} \log(\text{Set}_{\mathcal{D}_v}(x))$. This extension will be part of a longer version of the paper. Two examples of such families are graphs with degree parities ([Read and Robinson \(1982\)](#)), and graphs with a given degree sequence ([Bender and Canfield \(1978\)](#)).

We believe that complete asymptotic expansion can be derived for graphs with degree constraints. This would require to apply a more general version of [Lemma 2](#), such as presented in Chapter 4 by [Pemantle and Wilson \(2013\)](#), and we would have to consider more complex families than $\text{MG}_{n,m}^{(\mathcal{D},*)}$.

The asymptotics of connected graphs from $\text{SG}_{n,m}$ when $m - n$ tends to infinity has first been derived by [Bender et al. \(1990\)](#). Since then, two new proofs were given, one by [Pittel and Wormald \(2005\)](#), the other by [van der Hofstad and Spencer \(2006\)](#). The proof of Pittel and Wormald relies on a link between connected graphs and graphs from a particular family of graphs with degree constraints (graphs with degrees at least 2). In [de Panafieu \(2014\)](#), following the same approach, but using analytic combinatorics, we obtained a short proof for the asymptotics of connected multigraphs from $\text{MG}_{n,m}$ when $m - n$ tends to infinity. We now plan to extend this result to simple graphs, and to derive a complete asymptotic expansion.

In this paper, we have focused on the enumeration of graphs with degree constraints. We can now start the investigation on the typical structure of random instances of such graphs. An application would be the enumeration of Eulerian graphs, *i.e.* connected Euler graphs.

Finally, the inclusion-exclusion technique we used to remove loops and double edges can be extended to forbid any family of subgraphs.

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A Proofs

In this appendix, we include the proofs of the lemmas and theorems.

Proof of Theorem 3. By definition of the compensation factor, the number of multigraphs of the theorem is equal to

$$\frac{1}{2^m m!} \sum_{G \in \text{MG}_{n,m}^{(\mathcal{D})}} \text{orderings}(G).$$

Let us consider an ordering

$$((v_1, v_2), (v_2, v_3), \dots, (v_{2m-1}, v_{2m})).$$

of a multigraph G from $\text{MG}_{n,m}^{(\mathcal{D})}$. For all $1 \leq i \leq n$, let $P_i = \{j \mid v_j = i\}$ denote the set of positions of the vertex i in this ordering. Since the vertices have their degrees in \mathcal{D} , each P_i has size in \mathcal{D} . This implies a bijection between

- the orderings of multigraphs in $\text{MG}_{n,m}^{(\mathcal{D})}$,
- the sequences of sets (P_1, \dots, P_n) , where the size of each set is in \mathcal{D} , and (P_1, \dots, P_n) is a partition of $\{1, \dots, 2m\}$ (*i.e.* the sets are disjoint and $\bigcup_{i=1}^n P_i = \{1, \dots, 2m\}$).

We now interpret (P_1, \dots, P_n) as a sequence of sets that contain labelled objects and apply the *Symbolic Method* (see [Flajolet and Sedgewick \(2009\)](#)). The exponential generating function of sets of size in \mathcal{D} is $\text{Set}_{\mathcal{D}}(x)$. The bijection then implies

$$\sum_{G \in \text{MG}_{n,m}^{(\mathcal{D})}} \text{orderings}(G) = (2m)! [x^{2m}] \text{Set}_{\mathcal{D}}(x)^n,$$

and the theorem follows, after division by $2^m m!$. \square

Proof of Lemma 5. As explained in the paragraphs **First marking** and **Second marking** of Section 3.3.2, the following relations hold

$$\begin{aligned} \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},*)}}^{(1)}(0, 0) &= |\text{SG}_{n,m}^{(\mathcal{D})}|, \\ \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},*)}}^{(2)}(u, v) &= \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},*)}}^{(1)}(u+1, v+1). \end{aligned}$$

Furthermore, by construction of $\text{MG}_{n,m}^{(\mathcal{D},*)}$, we have

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},*)}}(u, v) = \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},*)}}^{(2)}(u, v),$$

so $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},*)}}(-1, -1) = |\text{SG}_{n,m}^{(\mathcal{D})}|$. \square

Proof of Lemma 6. To build an ordering of a multigraph from $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}$ with $2k$ vertices in marked double edges and ℓ vertices in marked loops, we perform the following steps:

1. choose the labels of the $2k$ vertices that appear in the marked double edges, and the ℓ vertices that appear in the marked loops. There are $\binom{n}{2k, \ell, n-2k-\ell}$ such choices.
2. choose the distinct k edges of distinct vertices, among the chosen $2k$ vertices, that will become the marked double edges. There are $\frac{(2k)!}{2^k k!}$ such choices.
3. order the $2k$ marked double edges and the vertices in each of them. There are $\frac{(2k)! 4^k}{2^k}$ ways to order them.

4. order the ℓ loops. There are $\ell!$ ways to do so.
5. choose among the m edges of the final ordering which ones receive marked loops and which ones receive marked double edges. There are $\binom{m}{2k, \ell, m-2k-\ell}$ choices.
6. to fill the rest of the final ordering, build an ordering of length $2m - 4k - 2\ell$ where the $2k$ vertices that belong to marked double edges and the ℓ vertices that appear in marked loops have degree in $\mathcal{D} - 2$, while the other $n - 2k - \ell$ vertices have degree in \mathcal{D} . The number of such orderings is $(2m - 4k - 2\ell)! [x^{2m-4k-2\ell}] \text{Set}_{\mathcal{D}-2}(x)^{2k+\ell} \text{Set}_{\mathcal{D}}(x)^{n-2k-\ell}$.

This bijective construction implies the following enumerative result

$$\begin{aligned}
& \sum_{G \in \text{Marked}(\text{MG}_{n,m}^{(\mathcal{D})})} \kappa(G) u^{k(G)} v^{\ell(G)} \\
&= \frac{1}{2^m m!} \sum_{k, \ell \geq 0} \binom{n}{2k, \ell, n-2k-\ell} \frac{(2k)!}{2^k k!} \frac{(2k)! 4^k}{2^k} \ell! \binom{m}{2k, \ell, m-2k-\ell} \\
&\quad (2m - 4k - 2\ell)! [x^{2m-4k-2\ell}] \text{Set}_{\mathcal{D}-2}(x)^{2k+\ell} \text{Set}_{\mathcal{D}}(x)^{n-2k-\ell} u^{k(G)} v^{\ell(G)}.
\end{aligned}$$

After simplification, this last expression can be rewritten

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(u, v) = \frac{(2m)!}{2^m m!} [x^{2m}] \left(\sum_{k, \ell \geq 0} a_{n,m,2k+\ell} \frac{(u W_{\frac{n}{m}}(x)^2)^k}{k!} \frac{(v W_{\frac{n}{m}}(x))^\ell}{\ell!} \right) \text{Set}_{\mathcal{D}}(x)^n.$$

□

Proof of Lemma 7. Developing the exponential as a double sum

$$e^{u+v} = \sum_{k, \ell \geq 0} \frac{u^k}{k!} \frac{v^\ell}{\ell!},$$

the result can be rewritten

$$n \frac{|1 - a_{n,m,2k+\ell}|}{k! \ell!} \leq [u^k v^\ell] C(u, v)$$

for all k, ℓ . We prove that when n is large enough, we have

$$n \frac{|1 - a_{n,m,2k+\ell}|}{k! \ell!} \leq \left(1 + \frac{n}{m}\right) \frac{(2k + \ell)^2 e^{4k+2\ell}}{\sqrt{k! \ell!}} \quad (4)$$

for all $k, \ell \geq 1$. Since the right-hand side are the coefficients of a function analytic on \mathbb{C}^2 , this will conclude the proof.

Let $b_{n,j}$ denote the value $\prod_{i=0}^{j-1} \left(1 - \frac{i}{n}\right)$, then observe that $a_{n,m,j}$ is equal to $b_{n,j} b_{m,j} / b_{2m,2j}$. Since $b_{n,j} \leq 1$, if $(c_{n,j})$ denotes a sequence such that $c_{n,j} \leq b_{n,j}$ for all (n, j) , then $c_{n,j} c_{m,j} \leq a_{n,m,j} \leq c_{2m,2j}^{-1}$, which implies

$$n \frac{|1 - a_{n,m,2k+\ell}|}{k! \ell!} \leq n \frac{\max(c_{2m,4k+2\ell}^{-1} - 1, 1 - c_{n,2k+\ell} c_{m,2k+\ell})}{k! \ell!}. \quad (5)$$

We now prove that Equation (4) holds both for $2k + \ell \leq \sqrt{m}/2$ and for $2k + \ell > \sqrt{m}/2$.

Case $2k + \ell \leq \sqrt{m}/2$. We prove by recurrence on j that $b_{n,j} \geq 1 - \frac{j^2}{n}$. The recurrence is initialized with $b_{n,0} = 1$. Assuming it is satisfied at rank j , then

$$b_{n,j+1} = \left(1 - \frac{j}{n}\right) b_{n,j} \geq \left(1 - \frac{j}{n}\right) \left(1 - \frac{j^2}{n}\right) \geq 1 - \frac{(j+1)^2}{n},$$

which concludes the proof of the recurrence. This implies, using Inequality (5),

$$n \frac{|1 - a_{n,m,2k+\ell}|}{k!\ell!} \leq \frac{n}{k!\ell!} \max \left(\frac{1}{1 - \frac{(4k+2\ell)^2}{2m}} - 1, 1 - \left(1 - \frac{(2k+\ell)^2}{n} \right) \left(1 - \frac{(2k+\ell)^2}{m} \right) \right).$$

Since $2k + \ell \leq \sqrt{m}/2$, the first argument of the maximum function is at most 1. The second argument is smaller than $(n^{-1} + m^{-1})(2k + \ell)^2$. Therefore, we have

$$n \frac{|1 - a_{n,m,2k+\ell}|}{k!\ell!} \leq \left(1 + \frac{n}{m} \right) \frac{(2k + \ell)^2}{k!\ell!},$$

and Inequality (4) is satisfied.

Case $2k + \ell > \sqrt{m}/2$. We first prove $b_{n,j} \geq e^{-j}$. To do so, we apply a sum-integral comparison in the expression

$$\log(b_{n,j}) = \sum_{i=0}^{j-1} \log \left(1 - \frac{i}{n} \right) \geq \int_0^j \log \left(1 - \frac{x}{n} \right) dx = -(n-j) \log \left(1 - \frac{j}{n} \right) - j \geq -j.$$

Inequality (5) then implies

$$n \frac{|1 - a_{n,m,2k+\ell}|}{k!\ell!} \leq \frac{n}{k!\ell!} \max \left(e^{4k+2\ell} - 1, 1 - e^{-(4k+2\ell)} \right) \leq \frac{n}{\sqrt{k!\ell!}} \frac{e^{4k+2\ell}}{\sqrt{k!\ell!}}.$$

We now prove that $n/\sqrt{k!\ell!}$ is smaller than 1 for n large enough. Indeed, $2k + \ell > \sqrt{m}/2$ implies $\max(k, \ell) \geq \sqrt{m}/8$, so

$$\frac{n}{\sqrt{k!\ell!}} \leq \frac{n}{\sqrt{\max(k, \ell)!}} \leq \frac{n}{(\sqrt{m}/8)!},$$

and since m/n stays in a compact interval of $\mathbb{R}_{>0}$, this last term tends to 0 with n . We then conclude

$$n \frac{|1 - a_{n,m,2k+\ell}|}{k!\ell!} \leq \frac{e^{4k+2\ell}}{\sqrt{k!\ell!}}$$

for n large enough, so Inequality (4) is satisfied. \square

Proof of Lemma 8. We start with the expression of $\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(u, v)$ derived in Lemma 6

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(u, v) = \frac{(2m)!}{2^m m!} [x^{2m}] \left(\sum_{k, \ell \geq 0} a_{n,m,2k+\ell} \frac{(uW_{\frac{n}{m}}(x)^2)^k}{k!} \frac{(vW_{\frac{n}{m}}(x))^\ell}{\ell!} \right) \text{Set}_{\mathcal{D}}(x)^n.$$

Using the notation

$$A(x) = e^{uW_{\frac{n}{m}}(x)^2 + vW_{\frac{n}{m}}(x)} - \sum_{k, \ell \geq 0} a_{n,m,2k+\ell} \frac{(uW_{\frac{n}{m}}(x)^2)^k}{k!} \frac{(vW_{\frac{n}{m}}(x))^\ell}{\ell!},$$

this implies

$$\frac{(2m)!}{2^m m!} [x^{2m}] e^{uW_{\frac{n}{m}}(x)^2 + vW_{\frac{n}{m}}(x)} \text{Set}_{\mathcal{D}}(x)^n - \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(u, v) = \frac{(2m)!}{2^m m!} [x^{2m}] A(x) \text{Set}_{\mathcal{D}}(x)^n.$$

Observe that $W_{\frac{n}{m}}(x)$ has valuation 0 and period p . According to Lemma 2, we have

$$\frac{(2m)!}{2^m m!} [x^{2m}] e^{uW_{\frac{n}{m}}(x)^2 + vW_{\frac{n}{m}}(x)} \text{Set}_{\mathcal{D}}(x)^n = \frac{(2m)!}{2^m m!} \frac{\text{Set}_{\mathcal{D}}(\zeta)^n}{\zeta^{2m}} \frac{p}{\sqrt{2\pi n \zeta \phi'(\zeta)}} e^{uW_{\frac{n}{m}}(\zeta)^2 + vW_{\frac{n}{m}}(\zeta)} (1 + O(n^{-1})),$$

so the demonstration is complete if we prove

$$\frac{(2m)!}{2^m m!} [x^{2m}] A(x) \text{Set}_{\mathcal{D}}(x)^n = \frac{(2m)!}{2^m m!} O\left(n^{-1} \frac{\text{Set}_{\mathcal{D}}(\zeta)^n}{\zeta^{2m} \sqrt{n}}\right).$$

The Taylor coefficients of $W_{\frac{n}{m}}(x)$ need not be positive, so we introduce the entire function

$$\tilde{W}_{\frac{n}{m}}(x) = \sum_{n \geq 0} |[z^n] W_{\frac{n}{m}}(z)| x^n,$$

which dominate $W_{\frac{n}{m}}(x)$ coefficient-wise. By application of Lemma 7, $\frac{1}{n} C(u \tilde{W}_{\frac{n}{m}}(x)^2, v \tilde{W}_{\frac{n}{m}}(x))$ dominates coefficient-wise $A(x)$, and therefore

$$\left| \frac{(2m)!}{2^m m!} [x^{2m}] A(x) \text{Set}_{\mathcal{D}}(x)^n \right| \leq \frac{(2m)!}{2^m m!} [x^{2m}] \frac{1}{n} C(u \tilde{W}_{\frac{n}{m}}(x)^2, v \tilde{W}_{\frac{n}{m}}(x)) \text{Set}_{\mathcal{D}}(x)^n.$$

Finally, according to Lemma 2, we have

$$\frac{(2m)!}{2^m m!} [x^{2m}] \frac{1}{n} C(u \tilde{W}_{\frac{n}{m}}(x)^2, v \tilde{W}_{\frac{n}{m}}(x)) \text{Set}_{\mathcal{D}}(x)^n = \frac{(2m)!}{2^m m!} O\left(n^{-1} \frac{\text{Set}_{\mathcal{D}}(\zeta)^n}{\zeta^{2m} \sqrt{n}}\right).$$

□

Proof of Lemma 9. Let \tilde{R} denote the set of multigraphs from $\text{MG}_{n,m}^{(\mathcal{D})}$ with j distinguished mandatory edges

$$e_1 = \{v_1, v_2\}, \dots, e_j = \{v_{2j-1}, v_{2j}\}.$$

Given an ordering of a multigraph from R , we can distinguish the first occurrences of the mandatory edges, in order to obtain the ordering of a multigraph from \tilde{R} . Therefore, the number of orderings of multigraphs from R is at most equal to the number of orderings of multigraphs from \tilde{R} . Dividing by $2^m m!$, this implies

$$\sum_{G \in R} \kappa(G) \leq \sum_{G \in \tilde{R}} \kappa(G),$$

so $\text{Marked}_R(1, 1) \leq \text{Marked}_{\tilde{R}}(1, 1)$.

Let W denote the fixed set of vertices that appear in the mandatory edges, and for all $w \in W$, let d_w denote the number of occurrences of the vertex w in the mandatory edges

$$d_w = |\{i \mid v_i = w\}|.$$

Let also $G_{n,m}^{(d)}$ denote the set of multigraphs with n vertices and m edges, where each vertex w from the mandatory edges has degree in $\mathcal{D} - d_w$ and the other vertices have degrees in \mathcal{D} . To construct an ordering from a multigraph in $\text{Marked}_{\tilde{R}}$, we choose the j positions of the mandatory edges among the m positions available, the order of the vertices in those edges, and mark or not each of them. Then the rest of the ordering is filled with an ordering from $\text{Marked}_{G_{n,m-j}^{(d)}}$. Therefore, the number of orderings from $\text{Marked}_{\tilde{R}}$ is at most

$$m^j 2^j 2^{m-j} (m-j)! \text{Marked}_{G_{n,m-j}^{(d)}}(1, 1).$$

Dividing by $2^m m!$ and using the fact that j is fixed, we obtain

$$\text{Marked}_{\tilde{R}}(1, 1) = O\left(\text{Marked}_{G_{n,m-j}^{(d)}}(1, 1)\right). \quad (6)$$

Following the steps of Lemma 6, $\text{Marked}_{G_{n,m-j}^{(d)}}(1, 1)$ is smaller than or equal to

$$\frac{(2m-2j)!}{2^{m-j} (m-j)!} [x^{2m-2j}] \left(\sum_{k, \ell \geq 0} a_{n,m-j,2k+\ell} \frac{W_{\frac{n}{m}}(x)^{2k+\ell}}{k! \ell!} \right) \left(\prod_{v \in W} \text{Set}_{\mathcal{D}-d_v}(x) \right) \text{Set}_{\mathcal{D}}(x)^{n-|W|}.$$

An application of the same argument as in the proof of Lemma 8 leads to

$$\text{Marked}_{G_{\mathcal{D}}^{(d)}(n,m-j)}(1,1) = \frac{(2m-2j)!}{2^{m-j}(m-j)!} O\left(\frac{\text{Set}_{\mathcal{D}}(\zeta)^{n-|W|}}{\zeta^{2(m-j)}\sqrt{n-|W|}}\right).$$

Since $|W|$ and j are fixed, this implies, using Lemma 8,

$$\text{Marked}_{G_{n,m-j}^{(d)}}(1,1) = \frac{(2m-2j)!}{2^{m-j}(m-j)!} \frac{2^m m!}{(2m)!} O\left(\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(1,1)\right).$$

Simplifying and injecting this relation from Equation (6), we obtain

$$\text{Marked}_{\tilde{R}}(1,1) = O(n^{-j} \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(1,1)).$$

□

Proof of Lemma 10. By definition, a multigraph G belongs to $\text{MG}_{n,m}^{(\mathcal{D},0)}$ if and only if it contains a vertex v that is in one of the following configurations:

1. the loop $\{v,v\}$ appears at least twice in $E(G)$,
2. there is a vertex u such that the edge $\{u,v\}$ appears at least three times,
3. there is a vertex u such that $\{v,v\}$ is in $E(G)$ and $\{u,v\}$ appears at least twice,
4. there are vertices u and w such that $\{u,v\}$ and $\{v,w\}$ both appear at least twice.

Let \tilde{R}_1 (resp. $\tilde{R}_2, \tilde{R}_3, \tilde{R}_4$) denote the set of multigraphs from $\text{MG}_{n,m}^{(\mathcal{D})}$ that contain a vertex in configuration 1 (resp. 2, 3, 4). We then have

$$\text{MG}_{n,m}^{(\mathcal{D},0)} = \tilde{R}_1 \cup \tilde{R}_2 \cup \tilde{R}_3 \cup \tilde{R}_4.$$

Let also R_1, R_2, R_3 and R_4 denote four subsets of $\text{MG}_{n,m}^{(\mathcal{D})}$, such that

1. the multigraphs from R_1 contain two occurrences of the loop $\{1,1\}$,
2. the multigraphs from R_2 contain three occurrences of the edge $\{1,2\}$,
3. the multigraphs from R_3 contain an occurrence of $\{1,1\}$ and two occurrences of $\{1,2\}$,
4. the multigraphs from R_4 contain two occurrences of $\{1,2\}$ and two occurrences of $\{1,3\}$

(see Figure 1). Given the symmetric roles of the vertices, the number of orderings from multigraphs in \tilde{R}_1 (resp. $\tilde{R}_2, \tilde{R}_3, \tilde{R}_4$) is lesser than or equal to n times (resp. n^2, n^2, n^3) the number of orderings from multigraphs in R_1 (resp. R_2, R_3, R_4). This implies

$$\begin{aligned} \text{Marked}_{\tilde{R}_1}(1,1) &\leq n \text{Marked}_{R_1}(1,1), \\ \text{Marked}_{\tilde{R}_2}(1,1) &\leq n^2 \text{Marked}_{R_2}(1,1), \\ \text{Marked}_{\tilde{R}_3}(1,1) &\leq n^2 \text{Marked}_{R_3}(1,1), \\ \text{Marked}_{\tilde{R}_4}(1,1) &\leq n^3 \text{Marked}_{R_4}(1,1), \end{aligned}$$

so

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},0)}}(1,1) \leq n \text{Marked}_{R_1}(1,1) + n^2 \text{Marked}_{R_2}(1,1) + n^2 \text{Marked}_{R_3}(1,1) + n^3 \text{Marked}_{R_4}(1,1).$$

The multigraphs from R_1 (resp. R_2, R_3, R_4) have 2 mandatory edges (resp. 3, 3, 4). Four applications of Lemma 9 lead to

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D},0)}}(1,1) = O(n^{-1}) \text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(1,1).$$

Finally, according to Lemma 8,

$$\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(1,1) = O\left(\text{Marked}_{\text{MG}_{n,m}^{(\mathcal{D})}}(-1,-1)\right).$$

□